1 The Laplace Transform

1.1 Definition

$$F(s) = \mathcal{L}\{f(t)\} = \lim_{\substack{\epsilon \to 0 \\ \tau \to \infty}} \int_{\epsilon}^{\tau} e^{-st} f(t)\, dt$$

1.2 Theorems

1. $\mathcal{L}$ linear

$$\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

2. $\mathcal{L}^{-1}$ linear

$$\mathcal{L}^{-1}\{\alpha_1 F_1(s) + \alpha_2 F_2(s)\} = \alpha_1 f_1(t) + \alpha_2 f_2(t)$$

3. Differentiation

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$$

4. Integration

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} F(s)$$

5. Initial value

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

6. Steady state value

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

7. Time scaling

$$\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\} = aF(as)$$

8. Frequency scaling

$$\mathcal{L}^{-1}\left\{F\left(\frac{s}{a}\right)\right\} = af(at)$$

9. Time delay

$$\mathcal{L}\{f(t-T)\} = e^{-sT} F(s)$$

10. Damping

$$\mathcal{L}^{-1}\{F(s+a)\} = e^{-at} f(t)$$

11. Product

$$\mathcal{L}\{f_1(t) f_2(t)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F_1(\sigma) F_2(s-\sigma)\, d\sigma$$

12. Convolution

$$\mathcal{L}\left\{\int_0^t f_1(\tau) f_2(t-\tau)\, d\tau\right\} = F_1(s) F_2(s)$$

1.3 Transform table

$F(s)$	$\mathcal{L}^{-1}\{F(s)\}$	$\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$
1	$\delta(t)$	$\sigma(t)=1$
$\frac{1}{s}$	$\sigma(t)=1$	t
$\frac{1}{s^2}$	t	$\frac{t^2}{2}$
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$	$\frac{t^n}{n!}$
$\frac{1}{(s+a)^n}$	$\frac{t^{n-1}}{(n-1)!}e^{-at}$	$\frac{1}{a^n}\left\{1-\left(1+at+\ldots+\frac{a^{n-1}t^{n-1}}{(n-1)!}\right)e^{-at}\right\}$
$\frac{1}{s+a}$	e^{-at}	$\frac{1}{a}\left(1-e^{-at}\right)$
$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-bt}-e^{-at}}{a-b}$	$\frac{1}{ab}\left(1-\frac{be^{-at}-ae^{-bt}}{b-a}\right)$
$\frac{s+z}{(s+a)(s+b)}$	$\frac{(z-a)e^{-at}-(z-b)e^{-bt}}{b-a}$	$\frac{z}{ab}\left(1-\frac{b(z-a)e^{-at}-a(z-b)e^{-bt}}{z(b-a)}\right)$
$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{1}{a^2}\left(1-(1+at)e^{-at}\right)$
$\frac{s+z}{(s+a)^2}$	$(1+(z-a)t)e^{-at}$	$\frac{z}{a^2}\left(1-(1-\frac{a}{z}(a-z)t)e^{-at}\right)$
$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin(at)$	$\frac{1}{a^2}(1-\cos(at))$
$\frac{s}{s^2+a^2}$	$\cos(at)$	$\frac{1}{a}\sin(at)$
$\frac{s+z}{s^2+a^2}$	$\frac{\sqrt{z^2+a^2}}{a}\sin(at+\varphi)$ $\varphi=\arctan(a/z),\quad z\geq 0$ $\varphi=\arctan(a/z)+\pi,\quad z<0$	$\frac{z}{a^2}-\frac{\sqrt{z^2+a^2}}{a^2}\cos(at+\varphi)$ $\varphi=\arctan(a/z),\quad z\geq 0$ $\varphi=\arctan(a/z)+\pi,\quad z<0$
$\frac{1}{(s+a)^2+b^2}$	$\frac{1}{b}e^{-at}\sin(bt)$	$\frac{1}{a^2+b^2}\left\{1-Ce^{-at}\sin(bt+\varphi)\right\}$ $C=\frac{\sqrt{a^2+b^2}}{b}$ $\varphi=\arcsin(1/C),\quad a\geq 0$ $\varphi=\pi-\arcsin(1/C),\quad a<0$
$\frac{s+z}{(s+a)^2+b^2}$	$\frac{1}{b}\sqrt{(z-a)^2+b^2}e^{-at}\sin(bt+\varphi)$ $\varphi=\arctan\left(\frac{b}{z-a}\right), z-a\geq 0$ $\varphi=\arctan\left(\frac{b}{z-a}\right)+\pi, z-a<0$	$\frac{z}{a^2+b^2}\left\{1-Ce^{-at}\sin(bt+\varphi)\right\}$ $C=\frac{\sqrt{(z-a)^2+b^2}}{\lvert z\rvert}\frac{\sqrt{a^2+b^2}}{b}$ $\varphi=\arcsin(1/C),\quad a-\frac{a^2+b^2}{z}\geq 0$ $\varphi=\pi-\arcsin(1/C),\quad a-\frac{a^2+b^2}{z}<0$
$\begin{cases}\frac{1}{s^2+2\zeta\omega_0 s+\omega_0^2}\\ \frac{s+z}{s^2+2\zeta\omega_0 s+\omega_0^2}\end{cases}$	Compute $\begin{cases}a=\zeta\omega_0\\ b=\omega_0\sqrt{1-\zeta^2}\end{cases}$ and substitute in the formulas above.	

Notes on the derivative formula at $t = 0$

The formula $\mathcal{L}(f') = sF(s) - f(0_-)$ must be interpreted very carefully when f has a discontinuity at $t = 0$. We'll give two examples of the correct interpretation.

First, suppose that f is the constant 1, and has no discontinuity at $t = 0$. In other words, f is the constant function with value 1. Then we have $f' = 0$, and $f(0_-) = 1$ (since there is no jump in f at $t = 0$). Now let's apply the derivative formula above. We have $F(s) = 1/s$, so the formula reads

$$\mathcal{L}(f') = 0 = sF(s) - 1$$

which is correct.

Now, let's suppose that g is a unit step function, *i.e.*, $g(t) = 1$ for $t > 0$, and $g(0) = 0$. In contrast to f above, g has a jump at $t = 0$. In this case, $g' = \delta$, and $g(0_-) = 0$. Now let's apply the derivative formula above. We have $G(s) = 1/s$ (exactly the same as F!), so the formula reads

$$\mathcal{L}(g') = 1 = sG(s) - 0$$

which again is correct.

In these two examples the functions f and g are the same except at $t = 0$, so they have the same Laplace transform. In the first case, f has no jump at $t = 0$, while in the second case g does. As a result, f' has no impulsive term at $t = 0$, whereas g does. As long as you keep track of whether your function has, or doesn't have, a jump at $t = 0$, and apply the formula consistently, everything will work out.